# Proofs of Some Formulae for the Determination of Deduction Rates for Early Retirement Schemes

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**Abstract**

In this note, we provide proofs of the statements made by the author in a previous publication regarding the determination of the deduction rates necessary to neutralize the long-term impact of early retirement schemes on the finances of a pension system in a steady state.

## 1 Introduction

The aim of this note is to prove the statements made by the author in Seyed Hosseini (2023, Section 6) regarding the necessary deduction rates for the neutralization of impacts of early retirement schemes on pension systems in a steady state. We thus provide a proof and a slight generalization of statements previously proven by Knell (2021, pp. S6–S10 in the Supplementary Appendix). The main difference between our results and those of Knell (2021) is that we provide direct proofs for point-based pension systems (as in Germany) and also consider a scenario in which early retirees continue to work until the reach the regular retirement age while receiving early pension payments. Taking a closer look at this scenario enables us to account for a recent change in pension regulations in Germany removing earnings limits for early retirees. To ease comparability, we use the same notation as Knell (2021) wherever appropriate.

This note is organized as follows: The second section introduces a model of a point-based pension system without an early retirement option. The third section modifies this model to allow for early retirement without the possibility for those receiving early pensions to continue working, and derives formulae for computing the necessary deduction rates. The fourth section considers a new model with early retirement. In this model, all early retirees are assumed to continue working while receiving a pension. Again, formulae are derived to calculate the necessary deduction rates to neutralize the impact of early retirement on pension finances. The note concludes with a brief summary of the results.

## 2 Base Model with Arbitrary Survivorship

We suppose that everyone starts working at age $A$, retires at age $R^*$ if they have not already died and dies at the latest at age $\omega$. The following notation will be used throughout this section:

$$\begin{aligned}
N(s) &:= \text{Instantaneous birth rate at time } s \\
W(t) &:= \text{Wage at time } t \\
\rho &:= \text{Replacement Rate} \\
b(t) &:= \text{Contribution rate at time } t \\
E(t) &:= \text{Total expenditure at time } t \\
C(t) &:= \text{Total contribution at time } t \\
S(a) &:= \text{Probability of survival up to age } a \\
P(t,s) &:= \text{Pension at time t of individuals born at time } s.
\end{aligned}$$

The total population of individuals older than $A$ at time $t$ will be denoted by $Q(t)$ and is denoted by

$$\int_A^\omega N(t-a)S(a)\mathrm{d}a.$$

The retired and the active population at time $t$ will be denoted by $M(t)$ and $L(t)$ and are given by

$$\begin{aligned}
M(t) &= \int_{R^*}^\omega N(t-a)S(a)\mathrm{d}a \\
L(t) &= \int_A^{R^*} N(t-a)S(a)\mathrm{d}a
\end{aligned}$$

We thus have

$$\begin{aligned}
E(t) &= \int_{R^*}^\omega N(t-a)P(t,t-a)S(a)\mathrm{d}a \\
C(t) &= b(t)W(t)L(t) = b(t)W(t)\int_A^{R^*} N(t-a)S(a)\mathrm{d}a.
\end{aligned}$$

The contribution rate at time $t$ is given by

$$b(t) = \frac{E(t)}{W(t)L(t)} = \frac{\int_{R^*}^\omega N(t-a)P(t,t-a)S(a)\mathrm{d}a}{W(t)\int_A^{R^*} N(t-a)S(a)\mathrm{d}a}$$

We denote by $\delta(s)$ the internal rate of return of the cohort of individuals born at time $s$; i. e. $\delta(s)$ satisfies the equation

$$\int_{s+A}^{s+R^*} b(t)W(t)\mathrm{e}^{-\delta(s)(t-s-R^*)}S(t-s)\mathrm{d}t = \int_{s+R^*}^{s+\omega} P(t,s)\mathrm{e}^{-\delta(s)(t-s-R^*)}S(t-s)\mathrm{d}t. \qquad (\#)$$

In order to proceed, we suppose that $N(s) = N_0\mathrm{e}^{\alpha s}$, $W(t) = W_0\mathrm{e}^{\beta t}$ and that $P(t,s) = \rho W(s+R^*)e^{\gamma(t-s-R^*)}$. We then have

$$\begin{aligned}
E(t) &= \int_{R^*}^\omega N_0 e^{\alpha(t-a)}\rho W(t-a+R^*)e^{\gamma(t-(t-a+R^*))}S(a)\mathrm{d}a \\
&= \int_{R^*}^\omega N_0 e^{\alpha(t-a)}\rho W_0 e^{\beta(t-a+R^*)}e^{\gamma(a-R^*)}S(a)\mathrm{d}a \\
&= \rho N_0 W_0 e^{t(\alpha+\beta)+R^*(\beta-\gamma)}\int_{R^*}^\omega e^{a(\gamma-\alpha-\beta)}S(a)\mathrm{d}a
\end{aligned}$$

and

$$W(t)L(t) = N_0 W_0 e^{(\alpha+\beta)t} \int_A^{R^*} \mathrm{e}^{-a\alpha} S(a)\mathrm{d}a$$

and therefore

$$b(t) = \rho \frac{e^{R^*(\beta-\gamma)} \int_{R^*}^{\omega} e^{a(\gamma-\alpha-\beta)} S(a)\mathrm{d}a}{\int_A^{R^*} \mathrm{e}^{-a\alpha} S(a)\mathrm{d}a} =: b$$

Plugging the latter expression for $b(t)$ into the left hand side of Equation (#) and computing we see that the left hand side is equal to

$$\rho W_0 \frac{e^{R^*(\beta-\gamma+\delta(s))+\beta s} \int_{R^*}^{\omega} e^{a(\gamma-\alpha-\beta)} S(a)\mathrm{d}a}{\int_A^{R^*} \mathrm{e}^{-a\alpha} S(a)\mathrm{d}a} \int_A^{R^*} \mathrm{e}^{(\beta-\delta(s))t} S(t)\mathrm{d}t.$$

The right hand side of Equation (#) is equal to

$$\rho W_0 e^{(\beta-\gamma+\delta(s))R^*+\beta s} \int_{R^*}^{\omega} e^{(\gamma-\delta(s))t} S(t)\mathrm{d}t.$$

Equation (#) is equivalent to

$$\frac{\int_{R^*}^{\omega} e^{a(\gamma-\alpha-\beta)} S(a)\mathrm{d}a}{\int_A^{R^*} \mathrm{e}^{-a\alpha} S(a)\mathrm{d}a} = \frac{\int_{R^*}^{\omega} e^{(\gamma-\delta(s))t} S(t)\mathrm{d}t}{\int_A^{R^*} \mathrm{e}^{(\beta-\delta(s))t} S(t)\mathrm{d}t}.$$

The latter equation is obviously solved by $\delta(s) = \alpha + \beta$. Furthermore, this is the unique solution of Equation (#) as the left hand side is a strictly increasing and the right hand side is a strictly decreasing function of $\delta(s)$.

# 3 Model with an Early Retirement Option

We suppose that everyone starts working at age $A$, retires at age $R^*$ if they have not already died and dies at the latest at age $\omega$. The following notation will be used throughout this section:

$$\begin{aligned}
N(s) &:= \text{Instantaneous birth rate at time } s \\
W(t) &:= \text{Wage at time } t \\
\rho &:= \text{Replacement Rate} \\
\widetilde{b}(t) &:= \text{Contribution rate at time } t \\
\widetilde{E}(t) &:= \text{Total expenditure at time } t \\
\widetilde{C}(t) &:= \text{Total contribution at time } t \\
S(a) &:= \text{Probability of survival up to age } a \\
P(t,s,R) &:= \text{Pension at time t of individuals born at time } s \text{ and retiring at age } R.
\end{aligned}$$

The total population at time $t$ will be denoted by $Q(t)$ and is denoted by

$$\int_A^{\omega} N(t-a)S(a)\mathrm{d}a.$$

We denote by $f(R,s)$ the probability density function describing the proportion of those born at time $s$ who retire at age $R$. In the following, we will assume that $f$ does not depend on $s$.

We will assume that working is not allowed after retirement.

The retired and the active population at time $t$ will be denoted by $\widetilde{M}(t)$ and $\widetilde{L}(t)$ and are given by

$$\widetilde{M}(t) = \int_A^\omega f(R) \int_R^\omega N(t-a)S(a)\mathrm{d}a\mathrm{d}R$$
$$\widetilde{L}(t) = Q(t) - \widetilde{M}(t) = \int_A^\omega N(t-R)S(R) - f(R) \int_R^\omega N(t-a)S(a)\mathrm{d}a\mathrm{d}R$$

We have

$$\widetilde{E}(t) = \int_A^\omega f(R) \int_R^\omega N(t-a)S(a)P(t,t-a,R)\mathrm{d}a\mathrm{d}R$$

and

$$\widetilde{C}(t) = \widetilde{b}(t)W(t)\widetilde{L}(t) = \widetilde{b}(t)W(t) \int_A^\omega N(t-R)S(R) - f(R) \int_R^\omega N(t-a)S(a)\mathrm{d}a\mathrm{d}R$$

The contribution rate at time $t$ is given by

$$\widetilde{b}(t) = \frac{\widetilde{E}(t)}{W(t)\widetilde{L}(t)} = \frac{\int_A^\omega f(R) \int_R^\omega N(t-a)S(a)P(t,t-a,R)\mathrm{d}a\mathrm{d}R}{W(t)\int_A^\omega N(t-R)S(R) - f(R) \int_R^\omega N(t-a)S(a)\mathrm{d}a\mathrm{d}R}$$

In order to proceed, we suppose that $N(s) = N_0\mathrm{e}^{\alpha s}$, $W(t) = W_0\mathrm{e}^{\beta t}$ and that $P(t,s,R) = Z(R)\rho\frac{R-A}{R^*-A}W(s+R)e^{\gamma(t-s-R)}$. We then have

$$\widetilde{E}(t) = \frac{\rho W_0 N_0 \mathrm{e}^{t(\alpha+\beta)}}{R^*-A} \int_A^\omega f(R)Z(R)(R-A)\mathrm{e}^{R(\beta-\gamma)} \int_R^\omega \mathrm{e}^{a(\gamma-\alpha-\beta)}S(a)\mathrm{d}a\mathrm{d}R$$
$$W(t)\widetilde{L}(t) = W_0N_0\mathrm{e}^{(\alpha+\beta)t} \int_A^\omega \mathrm{e}^{-\alpha R}S(R) - f(R) \int_R^\omega \mathrm{e}^{-\alpha a}S(a)\mathrm{d}a\mathrm{d}R$$

and therefore

$$\widetilde{b}(t) = \frac{\rho}{R^*-A} \frac{\int_A^\omega f(R)Z(R)(R-A)\mathrm{e}^{R(\beta-\gamma)} \int_R^\omega \mathrm{e}^{a(\gamma-\alpha-\beta)}S(a)\mathrm{d}a\mathrm{d}R}{\int_A^\omega \mathrm{e}^{-\alpha R}S(R) - f(R)\int_R^\omega \mathrm{e}^{-\alpha a}S(a)\mathrm{d}a\mathrm{d}R} =: \widetilde{b}$$

Assuming $\widetilde{b} = b$, we compute the access factor $Z(R)$ for which the internal rate of return of an individual planning to retire at age $R$ is equal to $\alpha+\beta$; i. e. we would like to choose $Z(R)$ such that the equation

$$\int_{s+A}^{s+R} bW(t)\mathrm{e}^{-(\alpha+\beta)(t-s-R)}S(t-s)\mathrm{d}t = \int_{s+R}^{s+\omega} P(t,s,R)\mathrm{e}^{-(\alpha+\beta)(t-s-R)}S(t-s)\mathrm{d}t. \quad (\#\#)$$

is satisfied. The left hand side of the latter equation is equal to

$$bW_0\mathrm{e}^{(\alpha+\beta)R+\beta s} \int_A^R \mathrm{e}^{-\alpha t}S(t)\mathrm{d}t$$

and its right hand side is equal to

$$Z(R)\rho\frac{R-A}{R^*-A}W_0\mathrm{e}^{(\alpha+\beta)R+\beta s+(\beta-\gamma)R} \int_R^\omega \mathrm{e}^{(\gamma-\alpha-\beta)t}S(t)\mathrm{d}t.$$

Thus, Equation (##) is solved by

$$Z(R) = \frac{R^* - A}{R - A}\frac{b}{\rho}\frac{\int_A^R \mathrm{e}^{-\alpha t} S(t)\mathrm{d}t}{\int_R^\omega \mathrm{e}^{(\gamma-\alpha-\beta)t} S(t)\mathrm{d}t}\mathrm{e}^{-(\beta-\gamma)R}. \tag{$*$}$$

Starting with Equation ($*$) and plugging the latter into the expression for $\widetilde{b}$ we obtain

$$\widetilde{b} = b\frac{\int_A^\omega f(R)\int_A^R \mathrm{e}^{-\alpha t} S(t)\mathrm{d}t\mathrm{d}R}{\int_A^\omega \mathrm{e}^{-\alpha R} S(R) - f(R)\int_R^\omega \mathrm{e}^{-\alpha a} S(a)\mathrm{d}a\mathrm{d}R}.$$

We claim that, $\widetilde{b} = b$. Indeed,

$$\begin{aligned}
\frac{\int_A^\omega f(R)\int_A^R \mathrm{e}^{-\alpha t} S(t)\mathrm{d}t\mathrm{d}R}{\int_A^\omega \mathrm{e}^{-\alpha R} S(R) - f(R)\int_R^\omega \mathrm{e}^{-\alpha a} S(a)\mathrm{d}a\mathrm{d}R} &= \frac{\int_A^\omega f(R)\int_A^R N_0\mathrm{e}^{-\alpha t} S(t)\mathrm{d}t\mathrm{d}R}{\int_A^\omega N_0\mathrm{e}^{-\alpha R} S(R) - f(R)\int_R^\omega N_0\mathrm{e}^{-\alpha a} S(a)\mathrm{d}a\mathrm{d}R} \\
&= \frac{\int_A^\omega f(R)\left(Q(0) - \int_R^\omega N_0\mathrm{e}^{-\alpha t} S(t)\mathrm{d}t\right)\mathrm{d}R}{Q(0) - \int_A^\omega f(R)\int_R^\omega N_0\mathrm{e}^{-\alpha a} S(a)\mathrm{d}a\mathrm{d}R} \\
&= \frac{Q(0) - \int_A^\omega f(R)\int_R^\omega N_0\mathrm{e}^{-\alpha a} S(a)\mathrm{d}a\mathrm{d}R}{Q(0) - \int_A^\omega f(R)\int_R^\omega N_0\mathrm{e}^{-\alpha a} S(a)\mathrm{d}a\mathrm{d}R} \\
&= 1
\end{aligned}$$

In other words, if the access factor $Z(R)$ is chosen to satisfy Equation (##), then the contribution rates in the pension systems with and without an early retirement scheme coincide.

**Proposition 3.1.** *The function $R \mapsto Z(R)$ defined using Equation ($*$) is the unique continuous function which guarantees the equality $b = \widetilde{b}$ for any probability distribution function $f$.*

**Remark 3.2.** It is easy to see that $Z(R^*) = 1$.

We choose $Z(R)$ as in Equation ($*$). Then

$$\begin{aligned}
0 &= \int_{s+A}^{s+R} bW(t)\mathrm{e}^{-(\alpha+\beta)(t-s-R)} S(t-s)\mathrm{d}t - \int_{s+R}^{s+\omega} P(t,s,R)\mathrm{e}^{-(\alpha+\beta)(t-s-R)} S(t-s)\mathrm{d}t \\
&= \int_{s+A}^{s+R^*} bW(t)\mathrm{e}^{-(\alpha+\beta)(t-s-R^*)} S(t-s)\mathrm{d}t - \int_{s+R^*}^{s+\omega} P(t,s,R^*)\mathrm{e}^{-(\alpha+\beta)(t-s-R^*)} S(t-s)\mathrm{d}t
\end{aligned}$$

Multiplying the second term in the equality with $\mathrm{e}^{(\alpha+\beta)(R^*-R)}$ we obtain

$$\begin{aligned}
0 &= \int_{s+A}^{s+R} bW(t)\mathrm{e}^{-(\alpha+\beta)(t-s-R^*)} S(t-s)\mathrm{d}t - \int_{s+R}^{s+\omega} P(t,s,R)\mathrm{e}^{-(\alpha+\beta)(t-s-R^*)} S(t-s)\mathrm{d}t \\
&= \int_{s+A}^{s+R^*} bW(t)\mathrm{e}^{-(\alpha+\beta)(t-s-R^*)} S(t-s)\mathrm{d}t - \int_{s+R^*}^{s+\omega} P(t,s,R^*)\mathrm{e}^{-(\alpha+\beta)(t-s-R^*)} S(t-s)\mathrm{d}t
\end{aligned}$$

and thus the equalities

$$\int_{s+R^*}^{s+\omega} P(t,s,R^*)\mathrm{e}^{-(\alpha+\beta)(t-s-R^*)}\frac{S(t-s)}{S(R)}\mathrm{d}t =$$
$$\int_{s+R}^{s+R^*} bW(t)\mathrm{e}^{-(\alpha+\beta)(t-s-R^*)}\frac{S(t-s)}{S(R)}\mathrm{d}t + \int_{s+R}^{s+\omega} P(t,s,R)\mathrm{e}^{-(\alpha+\beta)(t-s-R^*)}\frac{S(t-s)}{S(R)}\mathrm{d}t$$

and

$$\int_{s+R^*}^{s+\omega} P(t,s,R^*)\mathrm{e}^{-(\alpha+\beta)(t-s-R^*)}\frac{S(t-s)}{S(R^*)}\mathrm{d}t + \int_{s+R^*}^{s+R} bW(t)\mathrm{e}^{-(\alpha+\beta)(t-s-R^*)}\frac{S(t-s)}{S(R^*)}\mathrm{d}t$$
$$= \int_{s+R}^{s+\omega} P(t,s,R)\mathrm{e}^{-(\alpha+\beta)(t-s-R^*)}\frac{S(t-s)}{S(R^*)}\mathrm{d}t$$

for $R < R^*$ and $R^* < R$, respectively, which are often used as starting points for determining contribution-rate-stabilizing access factors.

# 4 Another Model with an Early Retirement Option

We suppose that everyone starts working at age $A$, retires at age $R^*$ if they are not already dead and dies at the latest at age $\omega$. The following notation will be used throughout this section:

$$\begin{aligned}
N(s) &\coloneqq \text{Instantaneous birth rate at time } s\\
W(t) &\coloneqq \text{Wage at time } t\\
\rho &\coloneqq \text{Replacement Rate}\\
\widehat{b}(t) &\coloneqq \text{Contribution rate at time } t\\
\widehat{E}(t) &\coloneqq \text{Total expenditure at time } t\\
\widehat{C}(t) &\coloneqq \text{Total contribution at time } t\\
S(a) &\coloneqq \text{Probability of survival up to age } a\\
\widehat{P}(t,s,R) &\coloneqq \text{Pension at time t of individuals born at time } s \text{ and retiring at age } R.
\end{aligned}$$

The total population at time $t$ will be denoted by $Q(t)$ and is denoted by

$$\int_A^\omega N(t-a)S(a)\mathrm{d}a.$$

We denote by $f(R,s)$ the probability density function describing the proportion of those born at time $s$ who retire at age $R$. In the following, we will assume that $f$ does not depend on $s$.

We will assume that individuals retiring at age $R < R^*$ continue to work until age $R^*$ and individuals retiring at age $R > R^*$ stop working.

The retired population at time $t$ will be denoted by $\widehat{M}(t)$ and is given by

$$\widehat{M}(t) = \int_A^\omega f(R)\int_R^\omega N(t-a)S(a)\mathrm{d}a\mathrm{d}R$$

The active population at time $t$ will be denoted by $\widehat{L}(t)$ and is given by

$$\begin{aligned}
\widehat{L}(t) =& Q(t) - \widehat{M}(t) + \int_A^{R^*} f(R)\int_R^{R^*} N(t-a)S(a)\mathrm{d}a\mathrm{d}R\\
=& \int_A^\omega N(t-R)S(R) - f(R)\int_R^\omega N(t-a)S(a)\mathrm{d}a\mathrm{d}R\\
&+ \int_A^{R^*} f(R)\int_R^{R^*} N(t-a)S(a)\mathrm{d}a\mathrm{d}R
\end{aligned}$$

We have

$$\widehat{E}(t) = \int_A^\omega f(R) \int_R^\omega N(t-a)S(a)\widehat{P}(t, t-a, R)\mathrm{d}a\mathrm{d}R$$

and

$$\widehat{C}(t) = \widehat{b}(t)W(t)\widehat{L}(t) = \widehat{b}(t)W(t)\Big( \int_A^\omega N(t-R)S(R) - f(R) \int_R^\omega N(t-a)S(a)\mathrm{d}a\mathrm{d}R \\ + \int_A^{R^*} f(R) \int_R^{R^*} N(t-a)S(a)\mathrm{d}a\mathrm{d}R \Big)$$

The contribution rate at time $t$ is given by

$$\widehat{b}(t) = \frac{\widehat{E}(t)}{W(t)\widehat{L}(t)}$$

In order to proceed, we suppose that

- $N(s) = N_0\mathrm{e}^{\alpha s}$,
- $W(t) = W_0\mathrm{e}^{\beta t}$ and
- $\widehat{P}(t,s,R) = Z(R)\rho\frac{R-A}{R^*-A}W(s+R)e^{\gamma(t-s-R)} + \rho\frac{\max(R^*-R,0)}{R^*-A}\overline{H}(t-s-R^*)W(s+R^*)e^{\gamma(t-s-R^*)}$,

where $\overline{H}$ is the step function defined by $H(x) = 0$ for all $x < 0$ and $\overline{H}(x) = 1$ for all $x \geq 0$. We then have

$$\widehat{E}(t) = \frac{\rho W_0 N_0 \mathrm{e}^{t(\alpha+\beta)}}{R^* - A} \Big( \int_A^\omega f(R)Z(R)(R-A)\mathrm{e}^{R(\beta-\gamma)} \int_R^\omega \mathrm{e}^{a(\gamma-\alpha-\beta)}S(a)\mathrm{d}a\mathrm{d}R \\ + \int_A^{R^*} f(R)(R^*-R)\mathrm{e}^{R^*(\beta-\gamma)} \int_{R^*}^\omega \mathrm{e}^{a(\gamma-\alpha-\beta)}S(a)\mathrm{d}a\mathrm{d}R \Big)$$

$$W(t)\widehat{L}(t) = W_0N_0\mathrm{e}^{t(\alpha+\beta)} \Big( \int_A^\omega \mathrm{e}^{-\alpha R}S(R) - f(R) \int_R^\omega \mathrm{e}^{-\alpha a}S(a)\mathrm{d}a\mathrm{d}R \\ + \int_A^{R^*} f(R) \int_R^{R^*} \mathrm{e}^{-\alpha a}S(a)\mathrm{d}a\mathrm{d}R \Big)$$

It follows that $\widehat{b}(t)$ is constant. We denote the constant by $\widehat{b}$.

Assuming $\widehat{b} = b$, we compute the access factor $Z(R)$ for which the internal rate of return of an individual planning to retire at age $R$ is equal to $\alpha + \beta$; i. e. we would like to choose $Z(R)$ such that the equation

$$\int_{s+A}^{s+R} bW(t)\mathrm{e}^{-(\alpha+\beta)(t-s-R)}S(t-s)\mathrm{d}t = \int_{s+R}^{s+\omega} \widehat{P}(t,s,R)\mathrm{e}^{-(\alpha+\beta)(t-s-R)}S(t-s)\mathrm{d}t. \quad (\#)$$

is satisfied for $R > R^*$ and the equation

$$\int_{s+A}^{s+R^*} bW(t)\mathrm{e}^{-(\alpha+\beta)(t-s-R)}S(t-s)\mathrm{d}t = \int_{s+R}^{s+\omega} \widehat{P}(t,s,R)\mathrm{e}^{-(\alpha+\beta)(t-s-R)}S(t-s)\mathrm{d}t. \quad (\#\#)$$

is satisfied for $R \leq R^*$.

The left hand side of Equation (#) is equal to

$$bW_0\mathrm{e}^{(\alpha+\beta)R+\beta s} \int_A^R \mathrm{e}^{-\alpha t}S(t)\mathrm{d}t$$

and its right hand side is equal to

$$Z(R)\rho\frac{R-A}{R^*-A}W_0\mathrm{e}^{(\alpha+\beta)R+\beta s+(\beta-\gamma)R}\int_R^\omega \mathrm{e}^{(\gamma-\alpha-\beta)t}S(t)\mathrm{d}t.$$

Thus, Equation (#) is solved by

$$Z(R)=\frac{R^*-A}{R-A}\frac{b}{\rho}\frac{\int_A^R \mathrm{e}^{-\alpha t}S(t)\mathrm{d}t}{\int_R^\omega \mathrm{e}^{(\gamma-\alpha-\beta)t}S(t)\mathrm{d}t}\mathrm{e}^{-(\beta-\gamma)R}. \tag{$*$}$$

The left hand side of Equation (##) is equal to

$$bW_0\mathrm{e}^{(\alpha+\beta)R+\beta s}\int_A^{R^*} \mathrm{e}^{-\alpha t}S(t)\mathrm{d}t$$

The right hand side of Equation (##) is equal to

$$\rho\frac{W_0 e^{(\alpha+\beta)R+\beta s}}{R^*-A}\Bigg(Z(R)(R-A)e^{(\beta-\gamma)R}\int_R^\omega \mathrm{e}^{(\gamma-\alpha-\beta)t}S(t)\mathrm{d}t \\ +(R^*-R)e^{(\beta-\gamma)R^*}\int_{R^*}^\omega \mathrm{e}^{(\gamma-\alpha-\beta)t}S(t)\mathrm{d}t\Bigg)$$

Thus, Equation (##) is solved by

$$Z(R)=\frac{R^*-A}{R-A}\frac{b}{\rho}\frac{\int_A^{R^*} \mathrm{e}^{-\alpha t}S(t)\mathrm{d}t}{\int_R^\omega \mathrm{e}^{(\gamma-\alpha-\beta)t}S(t)\mathrm{d}t}\mathrm{e}^{-(\beta-\gamma)R}-\frac{R^*-R}{R-A}\frac{\int_{R^*}^\omega \mathrm{e}^{(\gamma-\alpha-\beta)t}S(t)\mathrm{d}t}{\int_R^\omega \mathrm{e}^{(\gamma-\alpha-\beta)t}S(t)\mathrm{d}t}e^{(\beta-\gamma)(R^*-R)}. \tag{$**$}$$

Plugging $Z(R)$ as given by the formulas $(*)$ and $(**)$ into the definition of $\widehat{E}(t)$ and computing we obtain

$$W_0N_0\mathrm{e}^{t(\alpha+\beta)}\left(b\int_A^\omega f(R)\int_A^R \mathrm{e}^{-\alpha t}S(t)\mathrm{d}t\mathrm{d}R+b\int_A^{R^*} f(R)\int_R^{R^*} \mathrm{e}^{-\alpha t}S(t)\mathrm{d}t\mathrm{d}R\right)$$

Thus, with $Z(R)$ chosen as above $\widehat{b}$ is given by

$$\frac{b\int_A^\omega f(R)\int_A^R N_0\mathrm{e}^{-\alpha t}S(t)\mathrm{d}t\mathrm{d}R+b\int_A^{R^*} f(R)\int_R^{R^*} \mathrm{e}^{-\alpha t}S(t)\mathrm{d}t\mathrm{d}R}{\int_A^\omega \mathrm{e}^{-\alpha R}S(R)-f(R)\int_R^\omega \mathrm{e}^{-\alpha a}S(a)\mathrm{d}a\mathrm{d}R+\int_A^{R^*} f(R)\int_R^{R^*} \mathrm{e}^{-\alpha a}S(a)\mathrm{d}a\mathrm{d}R}$$

Using the equality

$$\int_A^\omega f(R)\int_A^R N_0\mathrm{e}^{-\alpha t}S(t)\mathrm{d}t\mathrm{d}R=\int_A^\omega \mathrm{e}^{-\alpha R}S(R)-f(R)\int_R^\omega \mathrm{e}^{-\alpha a}S(a)\mathrm{d}a\mathrm{d}R$$

proven in the previous section, it follows immediately that $b=\widehat{b}$. In other words, if the access factor $Z(R)$ is chosen to satisfy Equations (#) and (##), then the contribution rates in the pension systems with and without an early retirement scheme coincide.

**Proposition 4.1.** *The function $R\mapsto Z(R)$ defined using Equation $(*)$ is the unique continuous function which guarantees the equality $b=\widehat{b}$ for any probability distribution function $f$.*

**Remark 4.2.** It is easy to see that $Z(R^*)=1$.

We pick $Z(R)$ as in the formulas $(*)$ and $(**)$. Then for $R \leq R^*$

$$
\begin{aligned}
0 &= \int_{s+A}^{s+R^*} bW(t)\mathrm{e}^{-(\alpha+\beta)(t-s-R)}S(t-s)\mathrm{d}t - \int_{s+R}^{s+\omega} \widehat{P}(t,s,R)\mathrm{e}^{-(\alpha+\beta)(t-s-R)}S(t-s)\mathrm{d}t \\
&= \int_{s+A}^{s+R^*} bW(t)\mathrm{e}^{-(\alpha+\beta)(t-s-R^*)}S(t-s)\mathrm{d}t - \int_{s+R^*}^{s+\omega} \widehat{P}(t,s,R^*)\mathrm{e}^{-(\alpha+\beta)(t-s-R^*)}S(t-s)\mathrm{d}t
\end{aligned}
$$

Multiplying the second term in the equality with $\mathrm{e}^{(\alpha+\beta)(R^*-R)}$ we obtain

$$
\begin{aligned}
0 &= \int_{s+A}^{s+R^*} bW(t)\mathrm{e}^{-(\alpha+\beta)(t-s-R^*)}S(t-s)\mathrm{d}t - \int_{s+R}^{s+\omega} \widehat{P}(t,s,R)\mathrm{e}^{-(\alpha+\beta)(t-s-R^*)}S(t-s)\mathrm{d}t \\
&= \int_{s+A}^{s+R^*} bW(t)\mathrm{e}^{-(\alpha+\beta)(t-s-R^*)}S(t-s)\mathrm{d}t - \int_{s+R^*}^{s+\omega} \widehat{P}(t,s,R^*)\mathrm{e}^{-(\alpha+\beta)(t-s-R^*)}S(t-s)\mathrm{d}t
\end{aligned}
$$

and therefore the equality

$$
\int_{s+R}^{s+\omega} \widehat{P}(t,s,R)\mathrm{e}^{-(\alpha+\beta)(t-s-R^*)}S(t-s)\mathrm{d}t = \int_{s+R^*}^{s+\omega} \widehat{P}(t,s,R^*)\mathrm{e}^{-(\alpha+\beta)(t-s-R^*)}S(t-s)\mathrm{d}t.
$$

Using the decomposition

$$
\widehat{P}(t,s,R^*) = \rho\frac{R-A}{R^*-A}W(s+R^*)e^{\gamma(t-s-R^*)} + \rho\frac{R^*-R}{R^*-A}W(s+R^*)e^{\gamma(t-s-R^*)}
$$

and the definition of $\widehat{P}(t,s,R)$ we obtain the equality

$$
\begin{aligned}
&\int_{s+R}^{s+\omega} Z(R)\rho\frac{R-A}{R^*-A}W(s+R)e^{\gamma(t-s-R)}\mathrm{e}^{-(\alpha+\beta)(t-s-R^*)}S(t-s)\mathrm{d}t \\
&\qquad = \int_{s+R^*}^{s+\omega} \rho\frac{R-A}{R^*-A}W(s+R^*)e^{\gamma(t-s-R^*)}\mathrm{e}^{-(\alpha+\beta)(t-s-R^*)}S(t-s)\mathrm{d}t
\end{aligned}
$$

and therefore the equality

$$
\begin{aligned}
&\int_{s+R}^{s+\omega} Z(R)W(s+R)e^{\gamma(t-s-R)}\mathrm{e}^{-(\alpha+\beta)(t-s-R^*)}S(t-s)\mathrm{d}t \\
&\qquad = \int_{s+R^*}^{s+\omega} W(s+R^*)e^{\gamma(t-s-R^*)}\mathrm{e}^{-(\alpha+\beta)(t-s-R^*)}S(t-s)\mathrm{d}t
\end{aligned}
$$

which can be used as a starting point for the determination of $Z(R)$.

# 5 Summary

In this note we proved the statements made by the author in Seyed Hosseini (2023, Section 6). The results clarify the conditions under which the formulae often used in German-language literature can be employed to calculate deduction rates for early retirees that offset the impact of early retirement schemes on pension finances. The results also demonstrate how working while receiving early pensions can impact the computation of the necessary deduction rates.

# References

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